

The University of Chicago

ASYMPTOTIC GENERALIZATIONS OF WARING'S THEOREM

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1932

By
RALPH ERNEST HUSTON

Private Edition, Distributed by
THE UNIVERSITY OF CHICAGO LIBRARIES
CHICAGO, ILLINOIS

Reprinted from
PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY
Vol. 39 (1935).

The University of Chicago

ASYMPTOTIC GENERALIZATIONS OF WARING'S THEOREM

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1932

By

RALPH ERNEST HUSTON



Private Edition, Distributed by
THE UNIVERSITY OF CHICAGO LIBRARIES
CHICAGO, ILLINOIS

Reprinted from
PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY
Vol. 39 (1935).

ASYMPTOTIC GENERALIZATIONS OF WARING'S THEOREM*

By RALPH E. HUSTON.

[Received and read 16 March, 1933.—Revised 13 November, 1933.]

Contents.

	Page
Introduction	82
Notations	83
CHAPTER I. THE FIRST HARDY-LITTLEWOOD THEOREM EXTENDED.	
1. The integral expression for $r(n)$	84
2. Major and minor arcs	84
3. Lemmas for S , $S_{\nu p}$, $A_0(q)$, and $\mathfrak{E}_0$	85
4. Two lemmas for $f_\nu(x)$	87
5. On minor arcs, $f_\nu(x)$ approximates zero	88
6. On major arcs, $f_\nu(x)$ approximates $\psi_{\nu p}(x)$	88
7. The first Hardy-Littlewood theorem extended	92
CHAPTER II. THE SECOND HARDY-LITTLEWOOD THEOREM EXTENDED.	
8. Expression of $\mathfrak{E}_0$ as a function of χ_{0p}	98
9. A lower bound for χ_{0p} for all p 's	98
10. A lower bound for χ_{0p} for large p 's	101
11. Proof of Theorem 2... ..	104
CHAPTER III. SUFFICIENT CONDITIONS FOR WARING SETS OF COEFFICIENTS.	
12. Primitivity conditions	105
13. The case k an odd prime	107
14. Ordinary and special base sets	108
15. Derived coefficients	110
CHAPTER IV. SPECIAL CASES.	
16. The case $k=3$	110
17. Derived coefficients for $k=3$	111
18. The case $k=4$	112
19. The case $k=5$	114
Note on universal theorems	114

Introduction.

For every integral exponent $k \geq 3$ it is proposed to determine what number s of variables and what sets of positive integral coefficients

* A dissertation submitted for the degree of Ph.D. in the University of Chicago. The writer wishes to acknowledge his indebtedness to Prof. L. E. Dickson, who supervised his work as a graduate and who suggested this investigation.

$a_1, \dots, a_s$ ensure a solution of the diophantine equation

$$n = \sum_{\nu=1}^s a_{\nu} h_{\nu}^k \quad (h_{\nu} \geq 0)$$

for every integer $n > b_1$, where b_1 depends only on k, s , and $\max a_{\nu}$.

For $a_1 = \dots = a_s = 1$, the problem reduces to the classical Waring problem in asymptotic form. In this case the existence for every k of a sufficient s was proved by Hilbert* and the explicit value $s \geq (k-2)2^{k-1} + 5$ established by Hardy and Littlewood†. Our first aim will be to prove that this same value of s , if coupled with the congruential conditions of Theorem 2, is likewise sufficient in the more general problem. We shall then develop in the sections on primitivity and base sets important tests for determining when the hypotheses of Theorem 2 are satisfied. Finally we shall exhibit the results obtainable for the specific cases $k = 3, 4$, and 5 .

The method of proof throughout the chain of lemmas culminating in Theorems 1 and 2 follows as closely as possible that of Landau's presentation of the Hardy-Littlewood theory in his *Vorlesungen über Zahlentheorie* (hereafter quoted as Landau), erster Band, sechster Teil, Kapitel 1-2. In fact, Theorems 1 and 2 correspond respectively to what Landau calls the First and Second Hardy-Littlewood theorems. The principal ideas in the sections on primitivity, base sets, and derived coefficients have been developed by the writer.

Notation.

The notation for the four series of positive constants c_i, C_i, b_i, B_i denotes functional dependence as follows:

$$\begin{aligned} c(k, \max a_{\nu}), \quad C(k, \epsilon, \max a_{\nu}), \\ b(k, s, \max a_{\nu}), \quad B(k, s, \epsilon, \max a_{\nu}). \\ a = 1/k, \quad K = 2^{k-1}, \quad \kappa = 1 - 1/K. \end{aligned}$$

$$f_{\nu}(x) = \sum_{h=0}^{\infty} x^{a_{\nu} h^k}.$$

$$r(n) = \frac{1}{2\pi i} \int \prod_{\nu=1}^s f_{\nu}(x) \frac{dx}{x^{n+1}}.$$

Major arcs have $q \leq n^{\alpha}$.

Minor arcs have $n^{\alpha} < q \leq n^{1-\alpha}$.

$$S_{\nu\rho}(x) = \sum_h \rho^{a_{\nu} h^k},$$

* D. Hilbert, *Göttinger Nachrichten* (1909), 17; *Math. Annalen*, 67 (1909), 281.

† G. H. Hardy and J. E. Littlewood, *Math. Zeitschrift*, 12 (1922), 161.

where ρ is a primitive q -th root of unity and h ranges over a complete set of residues mod q .

$$A_0(q) = q^{-s} \sum_{\rho(q)} \prod_{\nu=1}^s S_{\nu\rho} \rho^{-n}.$$

$$\mathfrak{S}_0 = \sum_{q=1}^{\infty} A_0(q).$$

$$\psi_{\nu\rho}(x) = \frac{\Gamma(1+a)}{a_\nu^a} \frac{S_{\nu\rho}}{q} \frac{1}{(1-X)^a},$$

$$\chi_{0p} = 1 + A_0(p) + \dots = \sum_{l=0}^{\infty} A_0(p^l), \quad p \text{ prime.}$$

$$(a, b) = \text{g.c.d. } (a, b).$$

For $N_{sp}(p^l, n)$ and for P , see § 9.

CHAPTER I.

The first Hardy-Littlewood theorem extended.

1. *The integral expression for $r(n)$.* Let x be a complex variable and let

$$f_\nu(x) = \sum_{h=0}^{\infty} x^{a_\nu h k} \quad (\nu = 1, \dots, s).$$

The functions $f_\nu(x)$ and $\prod_{\nu=1}^s f_\nu(x)$ are regular for $|x| < 1$. If

$$r(n) = r(n, k, s, a_1, \dots, a_s)$$

denotes the number of solutions in non-negative integers of

$$(1) \quad n = \sum_{\nu=1}^s a_\nu h_\nu k,$$

then
$$\prod_{\nu=1}^s f_\nu(x) = \sum_{n=0}^{\infty} r(n) x^n,$$

and, by Cauchy's integral formula,

$$(2) \quad r(n) = \frac{1}{2\pi i} \int \prod_{\nu=1}^s f_\nu(x) \frac{dx}{x^{n+1}},$$

where the integral is taken in the positive direction round the circle $|x| = e^{-1/n}$. We seek sufficient conditions to be imposed on s and the coefficients a_ν to ensure that $r(n) > 0$ for every integer n sufficiently large. Henceforth let $n \geq 2$.

2. *Major and minor arcs on the circle of integration.* With Hardy-Littlewood and Landau let $\alpha = \alpha_s$ satisfy $0 < \alpha < 1$, its precise value being

fixed by (31') below. Consider on the interval $0 \leq l/q \leq 1$ those Farey fractions l/q which have $q \leq n^{1-\alpha}$. The corresponding $\rho = e^{2\pi i l/q}$ are then all primitive q -th roots of unity of index $q \leq n^{1-\alpha}$ with 1 occurring twice. Construct for each neighbouring pair, l/q and l'/q' , the point $M = (l+l')/(q+q')$ and subdivide the circle of integration into subarcs with end-points $x_M = e^{-1/n} e^{2\pi i M}$. Each subarc, since it contains exactly one of the points $e^{-1/n} e^{2\pi i l/q}$, is associated with a certain q -th root of unity ρ . If this $q \leq n^\alpha$, the subarc is a *major arc*; if $n^\alpha < q \leq n^{1-\alpha}$, it is a *minor arc*. The equations of both of these arcs may be written*

$$(3) \quad x = \rho X, \quad X = e^{-y}, \quad y = (1/n) - \theta i, \quad -\theta_1 \leq \theta \leq \theta_2,$$

where

$$(4) \quad \frac{\pi}{qn^{1-\alpha}} \leq \theta_1 \leq \frac{2\pi}{qn^{1-\alpha}}, \quad \frac{\pi}{qn^{1-\alpha}} \leq \theta_2 \leq \frac{2\pi}{qn^{1-\alpha}};$$

$$\theta_1 \leq \pi, \quad \theta_2 \leq \pi.$$

Landau proves in Satz 258 that, on every subarc,

$$(5) \quad |1-X| \geq c\sqrt{(n^{-2} + \theta^2)} > c|\theta|, \quad \frac{|1-X|}{1-|X|} \leq 1+n|\theta|,$$

where $c > 0$ is an absolute constant; that, on every minor arc,

$$(6) \quad \frac{|1-X|}{1-|X|} < 8;$$

and that, on every major arc,

$$(7) \quad \frac{|1-X|}{1-|X|} < 8n^\alpha/q.$$

For every n there will be a Farey subdivision and for each subdivision and on each subarc we shall find approximating functions $\psi_{\rho}(x)$ for the $f_{\nu}(x)$. Since these require different treatments on major and on minor arcs, we rewrite (2) in the form

$$(8) \quad r(n) = \frac{1}{2\pi i} \left(\sum_{\substack{\text{Major} \\ \text{arcs}}} \int_{Mj} \prod_{\nu=1}^s f_{\nu}(x) \frac{dx}{x^{n+1}} + \sum_{\substack{\text{Minor} \\ \text{arcs}}} \int_{Mi} \prod_{\nu=1}^s f_{\nu}(x) \frac{dx}{x^{n+1}} \right).$$

3. *Lemmas for S , $S_{\nu\rho}$, $A_0(q)$, and $\mathfrak{S}_0$.* Before introducing the $\psi_{\nu\rho}(x)$ we must establish a chain of lemmas.

* For these equations see Landau, 244; for details of the Farey subdivision, *ibid.*, 243-244, 98-100.

LEMMA 1. Let ρ be any primitive q -th root of unity, $m > 0$, let r be an integer, and let

$$S(a_\nu, \rho) = \sum_{h=r+1}^{r+m} \rho^{a_\nu h^k}.$$

Then, for every $\epsilon > 0$,

$$|S(a_\nu, \rho)|^K < C_{18} m^\epsilon q^\epsilon (m^{K-1} + m^K/q + m^{K-k}q).$$

For $a_\nu = 1$ the lemma becomes Landau's Satz 267. Denote by C'_{18} the value of C_{18} which serves here.

For $a_\nu > 1$ let $g = (a_\nu, q)$. Note that $1 \leq g \leq a_\nu \leq \max a_\nu$ and that $\rho^{a_\nu} = \rho'$, a primitive (q/g) -th root of unity. Hence, by the above case,

$$\begin{aligned} |S(a_\nu, \rho)|^K &= |S(1, \rho')|^K < C'_{18} m^\epsilon (q/g)^\epsilon (m^{K-1} + m^K g/q + m^{K-k} q/g) \\ &\leq C'_{18} m^\epsilon q^\epsilon (m^{K-1} + m^K g/q + m^{K-k} q) \\ &\leq C'_{18} g m^\epsilon q^\epsilon (m^{K-1} + m^K/q + m^{K-k} q) \\ &\leq C_{18} m^\epsilon q^\epsilon (m^{K-1} + m^K/q + m^{K-k} q) \end{aligned}$$

if $C'_{18} \max a_\nu = C_{18}$.

We obtain proofs of Lemmas 2-5 below by making verbal changes in the proofs of their analogues, Satz 268-Satz 271 in Landau; we replace S_ρ by $S_{\nu\rho}$, $A(q)$ by $A_0(q)$, $\mathfrak{S}$ by $\mathfrak{S}_0$, use Lemma 1 instead of Satz 267, and note that here the constants depend on $\max a_\nu$.

LEMMA 2. If, in the hypothesis of Lemma 1, $m \leq q$, then

$$|S(a_\nu, \rho)| < C_2 q^{\kappa+\epsilon}.$$

LEMMA 3. Let

$$S_{\nu\rho}(x) = \sum_h \rho^{a_\nu h^k},$$

where ρ is a primitive q -th root of unity and h ranges over a complete set of residues mod q . Then, for every $\epsilon > 0$,

$$|S_{\nu\rho}| < C_2 q^{\kappa+\epsilon}.$$

LEMMA 4. If $A_0(q) = q^{-s} \sum_{\rho(q)} \prod_{\nu=1}^s S_{\nu\rho} \rho^{-n}$,

then, for every integral $q > 0$, every $\epsilon > 0$, and every n ,

$$|A_0(q)| < B_1 q^{1-(s/K)+\epsilon}.$$

LEMMA 5. If $s > 2K$, the singular series

$$\mathfrak{S}_0 = \sum_{q=1}^{\infty} A_0(q)$$

converges absolutely for every n .

4. Two lemmas for $f_\nu(x)$.

LEMMA 6. If $\omega = [n^{a+\frac{1}{2}\epsilon}]$, then, on the entire circle $|x| = e^{-1/n}$, and for every $\epsilon > 0$,

$$\left| f_\nu(x) - \sum_{h=0}^{\omega} x^{a_\nu h^k} \right| < C_{33}.$$

The proof is as follows:

$$\begin{aligned} \left| f_\nu(x) - \sum_{h=0}^{\omega} x^{a_\nu h^k} \right| &= \left| \sum_{h=\omega+1}^{\infty} x^{a_\nu h^k} \right| \leq \sum_{h=\omega+1}^{\infty} |x|^{a_\nu h^k} \\ &= \sum_{h=\omega+1}^{\infty} (e^{-1/n})^{a_\nu h^k} \leq \sum_{h=\omega+1}^{\infty} e^{-h^k/n} < C_{33}, \end{aligned}$$

the last inequality being obtained in the course of the proof of Satz 274 in Landau.

LEMMA 7. Let ρ be an arbitrary root of unity, $j \geq 0$ an integer, $x = \rho X$, $\omega = [n^{a+\frac{1}{2}\epsilon}]$, and

$$\tau_\nu(j) = \sum_{h=0}^{[j^n/a_\nu a]} \rho^{a_\nu h^k} = \sum_{0 \leq a_\nu h^k \leq j} \rho^{a_\nu h^k};$$

then, on the entire circle $x = e^{-1/n}$,

$$(9) \quad f_\nu(x) = (1-X) \sum_{j=0}^{\infty} \tau_\nu(j) X^j,$$

and

$$(10) \quad \left| f_\nu(x) - (1-X) \sum_{j=0}^{a_\nu \omega^k} \tau_\nu(j) X^j - \tau_\nu(a_\nu \omega^k) X^{a_\nu \omega^k+1} \right| < C_{33}.$$

If $\tau_\nu(-1)$ is defined as zero, we have

$$\begin{aligned} f_\nu(x) &= \sum_{h=0}^{\infty} \rho^{a_\nu h^k} X^{a_\nu h^k} = \sum_{j=0}^{\infty} [\tau_\nu(j) - \tau_\nu(j-1)] X^j, \\ f_\nu(x) &= \sum_{j=0}^{\infty} \tau_\nu(j) (X^j - X^{j+1}) = (1-X) \sum_{j=0}^{\infty} \tau_\nu(j) X^j, \end{aligned}$$

which proves equation (9).

Secondly,

$$\begin{aligned} \sum_{h=0}^{\omega} x^{a_\nu h^k} &= \sum_{h=0}^{\omega} \rho^{a_\nu h^k} X^{a_\nu h^k} = \sum_{j=0}^{a_\nu \omega^k} [\tau_\nu(j) - \tau_\nu(j-1)] X^j \\ &= \sum_{j=0}^{a_\nu \omega^k} \tau_\nu(j) (X^j - X^{j+1}) + \tau_\nu(a_\nu \omega^k) X^{a_\nu \omega^k+1} \\ &= (1-X) \sum_{j=0}^{a_\nu \omega^k} \tau_\nu(j) X^j + \tau_\nu(a_\nu \omega^k) X^{a_\nu \omega^k+1}. \end{aligned}$$

Substitution of this value in Lemma 6 gives equation (10).

5. On minor arcs, $f_\nu(x)$ approximates zero.

LEMMA 8. On all minor arcs, for every $\epsilon > 0$,

$$|f_\nu(x)| < C_4 n^{a-(\alpha/K)+\epsilon}.$$

The proof is a straightforward generalization of the proof of Satz 276 in Landau: use $m = [j^a/a_\nu^a] + 1$ instead of $m = [j^a] + 1$, Lemma 1 instead of Satz 267, the $\tau_\nu(j)$ of Lemma 7 instead of $\tau(j)$, $0 \leq j \leq a_\nu \omega^k$ instead of $0 \leq j \leq \omega^k$, and (10) instead of (302).

6. On major arcs, $f_\nu(x)$ approximates $\psi_{\nu\rho}(x)$. For use in the proof of Lemma 11 we quote Lemmas 9 and 10 from Landau, where they are proved as Satz 277 and Satz 273, respectively.

LEMMA 9. For $\beta > 0$ and integral $j > 0$,

$$\left| \frac{\Gamma(\beta+1+j)}{j!} - j^\beta \right| < \gamma(\beta) j^{\beta-1},$$

where $\gamma = \gamma(\beta)$ depends only on β and not on j .

LEMMA 10. Let ρ be a primitive q -th root of unity,

$$q \leq n^a, \quad \epsilon > 0, \quad 1 \leq m \leq n^{a+\epsilon}, \quad \theta \text{ real},$$

$$\frac{1}{n} \left(\frac{n^a}{q} \right)^{K/(K+1)} \leq |\theta| \leq \frac{\pi}{k! q m^{k-1}},$$

$$S = \sum_{h=1}^m (\rho e^{i\theta})^{h^k};$$

then

$$|S| < C_{25} n^{aK/(K+1)+\epsilon}.$$

LEMMA 11. On major arcs, for every $\epsilon > 0$,

$$|f_\nu(x) - \psi_{\nu\rho}(x)| < C_5 n^{aK/(K+1)+\epsilon},$$

where $\psi_{\nu\rho}(x)$ is defined in equation (12) below.

[The proof falls into two cases if we put $(a_\nu, q) = d$, $q = dQ$, and divide the arc $\rho X = \rho e^{-1/n} e^{i\theta}$, $-\theta_1 \leq \theta \leq \theta_2$, into two parts according as

$$(I) \quad a_\nu |\theta| < \frac{1}{n} \left(\frac{n^a}{Q} \right)^{K/(K+1)},$$

$$(II) \quad a_\nu |\theta| \geq \frac{1}{n} \left(\frac{n^a}{Q} \right)^{K/(K+1)}.$$

Case I. By Lemma 7,

$$(9) \quad f_{\nu}(x) = (1-X) \sum_{j=0}^{\infty} \tau_{\nu}(j) X^j, \quad \tau_{\nu}(j) = \sum_{h=0}^{[j^a/a_{\nu}^a]} \rho^{a_{\nu} h^k}.$$

For every $j \geq 0$, $\tau_{\nu}(j)$ falls into $[j/a_{\nu}^a]$ partial sums of q terms each plus $[j^a/a_{\nu}^a] + 1 - [j^a/a_{\nu}^a q] q \leq q$ consecutive end-terms. Each of the partial sums is equal to $S_{\nu\rho}$. Lemma 2 applied to the end-terms yields

$$\begin{aligned} |\tau_{\nu}(j) - [j^a/a_{\nu}^a q] S_{\nu\rho}| &< C_2 q^{\kappa+\epsilon}, \\ |\tau_{\nu}(j) - (j^a/a_{\nu}^a q) S_{\nu\rho}| &< C_2 q^{\kappa+\epsilon} + |S_{\nu\rho}|. \end{aligned}$$

Since Lemma 3 ensures that $|S_{\nu\rho}| < C_2 q^{\kappa+\epsilon}$, this becomes

$$(11) \quad |\tau_{\nu}(j) - (j^a/a_{\nu}^a q) S_{\nu\rho}| < 2C_2 q^{\kappa+\epsilon} < 2C_2 q^{\kappa} n^{\epsilon}.$$

By definition,

$$(12) \quad \psi_{\nu\rho}(x) = (1-X) \frac{\Gamma(1+a)}{a_{\nu}^a} \frac{S_{\nu\rho}}{q} (1-X)^{-a-1} = (1-X) \sum_{j=0}^{\infty} v_{\nu}(j) X^j,$$

where

$$\begin{aligned} v_{\nu}(j) &= \frac{\Gamma(1+a)}{a_{\nu}^a} \frac{S_{\nu\rho}}{q} \binom{-a-1}{j} (-1)^j = \frac{S_{\nu\rho}}{a_{\nu}^a q} \Gamma(1+a) \frac{(a+1) \dots (a+j)}{j!}, \\ (13) \quad v_{\nu}(j) &= \frac{S_{\nu\rho}}{a_{\nu}^a q} \frac{\Gamma(a+1+j)}{j!}. \end{aligned}$$

By Lemma 9, for $j > 0$,

$$\left| \frac{\Gamma(a+1+j)}{j!} - j^a \right| < c_{20} j^{a-1} \leq c_{20};$$

hence, for $j \geq 0$,

$$(14) \quad \left| \frac{\Gamma(a+1+j)}{j!} - j^a \right| < c_{21}.$$

By (13), (14), and the fact that $|S_{\nu\rho}| \leq q$, we have, for $j \geq 0$,

$$(15) \quad \left| v_{\nu}(j) - \frac{j^a S_{\nu\rho}}{a_{\nu}^a q} \right| = \frac{1}{a_{\nu}^a} \left| \frac{S_{\nu\rho}}{q} \right| \cdot \left| \frac{\Gamma(a+1+j)}{j!} - j^a \right| < \frac{c_{21}}{a_{\nu}^a} \leq c_{21} < c_{21} q^{\kappa} n^{\epsilon}.$$

Now (11) and (15) give, for $j \geq 0$,

$$(16) \quad |\tau_{\nu}(j) - v_{\nu}(j)| < C_{39} q^{\kappa} n^{\epsilon}.$$

From (9), (12), (16), and the second equation of (5), it follows that

$$\begin{aligned}
 (17) \quad & |f_\nu(x) - \psi_{\nu\rho}(x)| \\
 &= \left| (1-X) \sum_{j=0}^{\infty} [\tau_\nu(j) - \nu_\nu(j)] X^j \right| < |1-X| C_{39} q^\kappa n^\epsilon \sum_{j=0}^{\infty} |X|^j \\
 &= C_{39} q^\kappa n^\epsilon \left| \frac{1-X}{1-|X|} \right| \leq C_{39} q^\kappa n^\epsilon (1+n|\theta|) \leq C_{39} q^\kappa n^\epsilon (1+na_\nu|\theta|).
 \end{aligned}$$

Since $Q \leq q \leq n^a < n^a$, condition (I) gives

$$1+na_\nu|\theta| < 1 + \left(\frac{n^a}{Q}\right)^{K/(K+1)} < 2 \left(\frac{n^a}{Q}\right)^{K/(K+1)}.$$

If we insert this in (17), substitute dQ for q , and use the relations $d \leq a_\nu \leq \max a_\nu$ and $Q \geq 1$, we obtain

$$\begin{aligned}
 |f_\nu(x) - \psi_{\nu\rho}(x)| &< C_{39} q^\kappa n^\epsilon 2 \left(\frac{n^a}{Q}\right)^{K/(K+1)} = 2C_{39} n^{aK/(K+1)+\epsilon} d^\kappa Q^\kappa Q^{-K/(K+1)} \\
 &\leq 2C_{39} n^{aK/(K+1)+\epsilon} (\max a_\nu)^\kappa Q^{\kappa-K/(K+1)} \leq C_{40} n^{aK/(K+1)+\epsilon},
 \end{aligned}$$

since the exponent of Q is negative.

Case II. Without loss of generality we may take $\epsilon < (a-a)/2k$, $\omega = [n^{a+\frac{1}{2}\epsilon}]$, and redefine

$$\tau_\nu(j) = \sum_{h \geq 0}^{0 \leq a_\nu h^k \leq j} \rho^{a_\nu h^k} e^{i\theta a_\nu h^k} = \sum_{h=0}^{[j^a/a_\nu^a]} \rho^{a_\nu h^k} e^{i\theta a_\nu h^k}.$$

If $a_\nu > j$,

$$(18) \quad \tau_\nu(j) = 1.$$

If $a_\nu \leq j \leq a_\nu \omega^k$, then $1 \leq j^a/a_\nu^a \leq \omega$. Since $\rho^{a_\nu} = \rho_1$ is a primitive Q -th root of unity,

$$\tau_\nu(j) - 1 = \sum_{h=1}^{[j^a/a_\nu^a]} \rho^{a_\nu h^k} e^{ia_\nu \theta h^k} = \sum_{h=1}^{[j^a/a_\nu^a]} (\rho_1 e^{a_\nu \theta i})^{h^k},$$

an expression which may be taken as the S of Lemma 10 with q replaced by Q , ρ by ρ_1 , θ by $\theta' = a_\nu \theta$, m by $m' = [j^a/a_\nu^a]$. We verify the hypotheses of Lemma 10, finding that $Q \leq q \leq n^a$, $1 \leq m' \leq j^a/a_\nu^a \leq \omega \leq n^{a+\frac{1}{2}\epsilon} < n^{a+\epsilon}$, and, by Case II,

$$\frac{1}{n} \left(\frac{n^a}{Q}\right)^{K/(K+1)} \leq |\alpha_r \theta| = |\theta'|.$$

The only hypothesis yet unverified is

$$|a_\nu \theta| \leq \frac{\pi}{k! Q m^{k-1}}.$$

This holds if $n > c_{24} = (k! 2 \max a_\nu)^{2/(a-a)}; \text{ for, with our } \epsilon, n > c_{24} \text{ implies that } n^{a-a-k\epsilon} > k! 2 \max a_\nu, \text{ and this coupled with } |\theta| a_\nu < 2\pi a_\nu / (qn^{1-a}), \text{ which follows from (4), ensures that}$

$$\frac{\pi}{k! Q m^{k-1}} \geq \frac{\pi}{k! q m^{k-1}} \geq \frac{\pi}{k! q n^{(k-1)a} n^{(k-1)\epsilon}} > \frac{\pi}{k! q n^{1-a} n^{k\epsilon}} = \frac{2\pi a_\nu n^{a-a-k\epsilon}}{q n^{1-a} k! 2a_\nu} > a_\nu |\theta|.$$

Hence, by Lemma 10, for $n > c_{24}$ and $a_\nu \leq j \leq a_\nu \omega^k$,

$$(19) \quad |\tau_\nu(j) - 1| < C_{25} n^{aK/(K+1)+\epsilon}.$$

For the subcase $n \leq c_{24}$, the proof of Lemma 11 is immediate:

$$(20) \quad |f_\nu(x) - \psi_{\nu\rho}(x)| \leq |f_\nu(x)| + |\psi_{\nu\rho}(x)| \\ \leq \sum_{h=0}^{\infty} e^{-a_\nu h^k/c_{24}} + a_\nu^{-a} \Gamma(1+a) (1 - e^{-1/c_{24}})^{-a} \\ \leq c_{\nu 25} \leq \max c_{\nu 25} < C_{41} n^{aK/(K+1)+\epsilon}.$$

For $n > c_{24}$ and $0 \leq j \leq a_\nu \omega^k$, we first derive, from (18) and (19),

$$(21) \quad |\tau_\nu(j)| < C_{42} n^{aK/(K+1)+\epsilon}.$$

Now by Lemma 6, with $R = e^{-1/n} = |x|$, we have

$$(22) \quad |f_\nu(x)| < C_{33} + \left| \sum_{h=0}^{\omega} x^{a_\nu h^k} \right| = C_{33} + \left| \sum_{h=0}^{\omega} \rho^{a_\nu h^k} e^{ia_\nu \theta h^k} R^{a_\nu h^k} \right|.$$

If we define $\tau_\nu(-1)$ as zero, we have

$$\sum_{h=0}^{\omega} \rho^{a_\nu h^k} e^{ia_\nu \theta h^k} R^{a_\nu h^k} = \sum_{j=0}^{a_\nu \omega^k} [\tau_\nu(j) - \tau_\nu(j-1)] R^j \\ = (1-R) \sum_{j=0}^{a_\nu \omega^k} \tau_\nu(j) R^j + \tau_\nu(a_\nu \omega^k) R^{a_\nu \omega^k + 1};$$

so that, by (21),

$$(23) \quad \left| \sum_{h=0}^{\omega} \rho^{a_\nu h^k} e^{ia_\nu \theta h^k} R^{a_\nu h^k} \right| \\ < C_{42} n^{aK/(K+1)+\epsilon} \left\{ (1-R) \sum_{j=0}^{\infty} R^j + 1 \right\} < C_{43} n^{aK/(K+1)+\epsilon}.$$

From (22) and (23) it follows that, for $n > c_{24}$,

$$(24) \quad |f_\nu(x)| < C_{44} n^{aK/(K+1)+\epsilon}.$$

By equation (5), $|1-X| \geq c|\theta|$. Hence Lemma 3 gives

$$\begin{aligned}
 (25) \quad |\psi_{\nu\rho}(x)| &= \frac{\Gamma(1+a)}{a_\nu^a} \left| \frac{S_{\nu\rho}}{q} \right| \frac{1}{|1-X|^a} < \frac{\Gamma(1+a)}{1} C_2 q^{-1/K} q^\epsilon \frac{1}{(c|\theta|)^a} \\
 &< C'_{45} q^{-1/K} n^\epsilon \left\{ a_\nu n \left(\frac{Q}{n^a} \right)^{K/(K+1)} \right\}^a \\
 &\leq C'_{45} (\max a_\nu)^a q^{-1/K} n^\epsilon \left\{ n \left(\frac{Q}{n^a} \right)^{K/(K+1)} \right\}^a \\
 &< C''_{45} n^{aK/(K+1)+\epsilon},
 \end{aligned}$$

where the last inequality is proved by Landau, 268.

Taken together, (24) and (25) yield, for $n > c_{24}$,

$$|f_\nu(x) - \psi_{\nu\rho}(x)| < C'_{44} n^{aK/(K+1)+\epsilon},$$

which with (20) completes the proof of Case II. Lemma 11 follows with $C_5 = C_{40} + C'_{44}$.

7. *The first Hardy-Littlewood theorem extended.* In order to avoid breaking the proof of Theorem 1, we first quote three lemmas from Landau and prove the supplementary Lemma 15.

LEMMA 12. (Landau, Satz 223.) If $r > 0$ and if

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

is regular for $|x| \leq r$, then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\phi})|^2 d\phi = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

LEMMA 13. (Landau, Satz 224.) If $0 < r < 1$ and $\epsilon > 0$, then

$$\sum_{m=1}^{\infty} m^\epsilon r^m < \frac{A}{(1-r)^{1+\epsilon}},$$

where A is positive and depends only on ϵ .

LEMMA 14. (Landau, Satz 262-1). The number of solutions $R(m) = R(m, k)$ of the diophantine equation

$$h_1^k + h_2^k = m \quad (h_1 \geq 0, h_2 \geq 0)$$

satisfies, for every $m > 0$ and every $\epsilon > 0$, the inequality

$$R(m) < C_{10} m^\epsilon.$$

LEMMA 15. (1) *The number of solutions $R_\nu(m) = R_\nu(m, a_\nu, k)$ of the diophantine equation*

$$a_\nu(h_1^k + h_2^k) = m \quad (h_1 \geq 0, h_2 \geq 0)$$

satisfies, for every $m > 0$ and every $\epsilon > 0$,

$$R_\nu(m) < C_{10} m^\epsilon.$$

(2) *For every $m > 0$,*

$$\sum_{j=0}^m R_\nu(j) \leq 4m^{2a}.$$

(3) *For $m > 0$,*

$$W_\nu(m) = \sum_{j=0}^m R_\nu^2(j) < C_{11} m^{2a+\epsilon}.$$

If $m \not\equiv 0 \pmod{a_\nu}$, $R_\nu(m) = 0$. If $m = m_\nu a_\nu$, then we have

$$R_\nu(m) = R(m_\nu) < C_{10} m_\nu^\epsilon \leq C_{10} m^\epsilon$$

by Lemma 14, proving (1).

Since j in (2) does not exceed m , $h_1 \leq m^a/a_\nu^a \leq m^a$ and $h_2 \leq m^a/a_\nu^a \leq m^a$. Hence the maximum number of values each can take is at most $1 + m^a \leq 2m^a$, and the maximum number of pairs is at most $4m^{2a}$. This proves (2).

From (1) and (2), for $m > 0$, relation (3) follows thus:

$$\sum_{j=0}^m R_\nu^2(j) \leq \max_{0 \leq j \leq m} R_\nu(j) \sum_{j=0}^m R_\nu(j) < C_{14} m^\epsilon 4m^{2a} = C_{11} m^{2a+\epsilon}.$$

We are now ready to construct the proof of Theorem 1, the first to involve explicitly the $r(n)$ of equations (2) and (8) about which our investigation began. Henceforth, unless otherwise indicated, let Π indicate the product for $\nu = 1, \dots, s$.

THEOREM 1. *For $s > (k-2)K+4$,*

$$(26) \quad \left| r(n) - \frac{\Gamma^s(1+a)}{\Gamma(sa) \prod a_\nu^a} n^{sa-1} \mathfrak{S}_0 \right| < b_7 n^{sa-1-c_{27}}.$$

This theorem is an extension for general positive integral coefficients of what Landau calls the first Hardy-Littlewood theorem.

Since, for $x = e^{-1/n} e^{i\theta}$, $|dx/x^{n+1}| = e|d\theta|$, equation (8) implies that

$$(27) \quad \left| r(n) - \frac{1}{2\pi i} \sum_{Mj} \int_{Mj} \Pi f_\nu(x) \frac{dx}{x^{n+1}} \right| = \left| \sum_{Mi} \int_{Mi} \Pi f_\nu(x) \frac{dx}{x^{n+1}} \right| \\ \leq \frac{e}{2\pi} \sum_{Mi} \int_{Mi} |\Pi f_\nu(x)| d\theta.$$

On minor arcs, ϵ/s instead of ϵ in Lemma 8 gives, for all values of ν ,

$$\begin{aligned}
 |f_\nu(x)| &< B_3 n^{a-\alpha/K+\epsilon/s}; \\
 \int_{Mi} \left| \prod_{\nu=1}^s f_\nu(x) \right| d\theta &\leq \left(\max_{Mi} \left| \prod_{\nu=5}^s f_\nu(x) \right| \right) \int_{Mi} \left| \prod_{\nu=1}^4 f_\nu(x) \right| d\theta \\
 &< B_3^{s-4} n^{(s-4)(a-\alpha/K)+\epsilon} \int_{Mi} \left| \prod_{\nu=1}^4 f_\nu(x) \right| d\theta, \\
 (28) \quad \sum_{Mi} \int_{Mi} \left| \prod_{\nu=1}^s f_\nu(x) \right| d\theta &\leq B_7 n^{(s-4)(a-\alpha/K)+\epsilon} \int_{-\pi}^{\pi} \left| \prod_{\nu=1}^4 f_\nu(e^{-1/n} e^{i\phi}) \right| d\phi \\
 &\leq B_7 n^{(s-4)(a-\alpha/K)+\epsilon} \int_{-\pi}^{\pi} \sum_{\nu=1}^4 |f_\nu^4(e^{-1/n} e^{i\phi})| d\phi.
 \end{aligned}$$

For $R_\nu(m)$, as in Lemma 15,

$$f_\nu^2(x) = \sum_{m=0}^{\infty} R_\nu(m) x^m,$$

to which Lemma 12 is applicable with $r = e^{-1/n}$, $f(x)$ replaced by $f_\nu^2(x)$, $a_n x^n$ by $R_\nu(m) x^m$, and $W_\nu(m)$ as in Lemma 15, (3). Hence, if

$$W_\nu(0) = R_\nu^2(0) = 1,$$

$$\begin{aligned}
 \int_{-\pi}^{\pi} |f_\nu^4(e^{-1/n} e^{i\phi})| d\phi &= 2\pi \sum_{m=0}^{\infty} R_\nu^2(m) e^{-2m/n} \\
 &= 2\pi \sum_{m=0}^{\infty} \{W_\nu(m) - W_\nu(m-1)\} e^{-2m/n} \\
 &= 2\pi \sum_{m=0}^{\infty} W_\nu(m) [e^{-2m/n} - e^{-2(m+1)/n}] \\
 &= 2\pi (1 - e^{-2/n}) \sum_{m=0}^{\infty} W_\nu(m) e^{-2m/n}.
 \end{aligned}$$

By Lemma 15, for $m > 0$, $W_\nu(m) < C_{11} m^{2a+\epsilon}$. Hence, by Lemma 13, with ϵ replaced by $2a+\epsilon$, r by $e^{-2/n}$, and A by C_{46} ,

$$\begin{aligned}
 (29) \quad \int_{-\pi}^{\pi} |f_\nu^4(e^{-1/n} e^{i\phi})| d\phi &< 2\pi (1 - e^{-2/n}) \{1 + \sum_{m=1}^{\infty} C_{11} m^{2a+\epsilon} (e^{-2/n})^m\} \\
 &< 2\pi (1 - e^{-2/n}) \{1 + C_{46} (1 - e^{-2/n})^{-1-2a-\epsilon}\} \\
 &< C_{47} n^{2a+\epsilon}.
 \end{aligned}$$

From (29) and (28),

$$(30) \quad \sum_{Mi} \int_{Mi} \left| \prod_{\nu=1}^s f_\nu(x) \right| d\theta < B_7^4 C_{47} n^{(s-4)(a-\alpha/K)+2a+2\epsilon}.$$

If we write $\frac{1}{2}\epsilon$ instead of ϵ and put $B_6 = 2eB_7C_{47}/\pi$, the inequality (30) reduces (27) to

$$(31) \quad \left| r(n) - \frac{1}{2\pi i} \sum_{Mj} \int_{Mj} \Pi f_v(x) \frac{dx}{x^{n+1}} \right| < B_6 n^{(s-4)(a-a/K)+2a+\epsilon}.$$

Throughout the rest of this section let $s > (k-2)K+4$; for $k=3$ let $a=17/60$; and for $k \geq 4$ let a satisfy

$$(31') \quad (1-2a)K/a+4 = (k-2)K+4.5.$$

Then, as shown in Landau, 271 and 246, for a suitable $\epsilon = c_{29}$,

$$(s-4)(a-a/K)+2a+\epsilon \leq sa-1-c_{29},$$

so that (31) becomes

$$(32) \quad \left| r(n) - \frac{1}{2\pi i} \sum_{Mj} \int_{Mj} \Pi f_v(x) \frac{dx}{x^{n+1}} \right| < b_8 n^{sa-1-c_{29}}.$$

Preparatory to replacing $f_v(x)$ by $\psi_{vp}(x)$, we examine

$$(33) \quad \left| \sum_{Mj} \int_{Mj} \Pi f_v(x) \frac{dx}{x^{n+1}} - \sum_{Mj} \int_{Mj} \Pi \psi_{vp}(x) \frac{dx}{x^{n+1}} \right| \\ \leq e \sum_{Mj} \int_{Mj} |\Pi f_v(x) - \Pi \psi_{vp}(x)| d\theta.$$

On major arcs Lemma 11, with ϵ/s instead of ϵ , gives, for every $\epsilon > 0$,

$$(34) \quad |f_v(x) - \psi_{vp}(x)| < B_8 n^{aK/(K+1)+\epsilon/s} = \Psi^*.$$

By an identity obtained from Taylor's expansion and by (34),

$$(35) \quad |\Pi f_v(x) - \Pi \psi_{vp}(x)| = \left| \sum_{\substack{\nu=1 \\ s \text{ terms}}}^s (f_v - \psi_\nu) \psi_\nu^{-1} \Pi \psi_j \right. \\ \left. + \sum_{\binom{s}{2}} \sum_{\substack{\nu \\ \mu}} (f_\nu - \psi_\nu)(f_\mu - \psi_\mu) \psi_\nu^{-1} \psi_\mu^{-1} \Pi \psi_j + \dots \right| \\ < \Psi^s \sum_{\nu} |\psi_\nu^{-1} \Pi \psi_j| + \Psi^{s-2} \sum_{\nu} \sum_{\mu} |\psi_\nu^{-1} \psi_\mu^{-1} \Pi \psi_j| + \dots + \Psi^s,$$

where all products are taken for ν or j from 1 to s . Let T_ν be equal to the last member of (35) with every $\psi_{jp} \neq \psi_{vp}$ replaced by ψ_{vp} ; i.e.

$$T_\nu = \Psi^s \left\{ s |\psi_\nu|^{s-1} + \binom{s}{2} \Psi |\psi_\nu|^{s-2} + \dots + \binom{s}{s-1} \Psi^{s-2} |\psi_\nu| + \Psi^{s-1} \right\},$$

$$(36) \quad T_\nu \leq \Psi^s 2^s \max(\Psi^{s-1}, |\psi_\nu|^{s-1}) \leq 2^s \{\Psi^s + \Psi |\psi_\nu|^{s-1}\},$$

since T_ν has 2^s terms. By (35) and (36), then (34),

$$\begin{aligned} |\Pi f_\nu(x) - \Pi \psi_{\nu\rho}(x)| &< \Sigma T_\nu \leq s 2^s \{B_8^s n^{aKs/(K+1)+\epsilon} + B_8 n^{aK/(K+1)+\epsilon} |\psi_{\nu\rho}|^{s-1}\}, \\ (37) \quad \Sigma \int_{Mj} |\Pi f_\nu(x) - \Pi \psi_{\nu\rho}(x)| d\theta \\ &< B_9 n^{aKs/(K+1)+\epsilon} \Sigma \int_{Mj} d\theta + B_9 n^{aK/(K+1)+\epsilon} \Sigma \int_{Mj} |\psi_{\nu\rho}(x)|^{s-1} d\theta. \end{aligned}$$

By Lemma 3, with $\epsilon/(s-1)$ instead of ϵ , and by the first of equations (5),

$$(38) \quad |\psi_{\nu\rho}(x)| = \frac{\Gamma(1+a)}{a_\nu^a} \left| \frac{S_{\nu\rho}}{q} \right| |1-X|^{-a} < B_{10} q^{-1/K} q^{\epsilon/(s-1)} (n^{-2} + \theta^2)^{-\frac{1}{2}a},$$

which is the same upper bound as that found by Landau, 272, for $|\psi_\rho(x)|$. Hence, by the argument given there and on 273, for $s > (k-2)K+4$ and for a suitable $\epsilon = c_{30}$,

$$(39) \quad \Sigma \int_{Mj} |\Pi f_\nu(x) - \Pi \psi_{\nu\rho}(x)| d\theta < b_9 n^{sa-1-c_{31}},$$

and, by (32) and (33),

$$(40) \quad \left| r(n) - \frac{1}{2\pi i} \Sigma \int_{Mj} \Pi \psi_{\nu\rho}(x) \frac{dx}{x^{n+1}} \right| < b_{10} n^{sa-1-c_{32}}.$$

The rest of the proof of Theorem 1 is closely parallel to the work of Landau. If in the *Zweiter Schritt* we use Lemma 3 instead of (272), we conclude that the error introduced by integrating in (40) around the entire circle C instead of merely along major arcs is expressed, for $s > (k-2)K+4 > 2K$ and for an appropriate $\epsilon = c_{33}$, by

$$(41) \quad \left| \Sigma \int_{Mj} \Pi \psi_{\nu\rho}(x) \frac{dx}{x^{n+1}} - \Sigma \int_C \Pi \psi_{\nu\rho}(x) \frac{dx}{x^{n+1}} \right| < b_{11} n^{sa-1-c_{34}};$$

so that, by (40) and (41),

$$(42) \quad \left| r(n) - \frac{1}{2\pi i} \Sigma \int_C \Pi \psi_{\nu\rho}(x) \frac{dx}{x^{n+1}} \right| < b_{12} n^{sa-1-c_{35}}.$$

To evaluate the integrals of (42), we write

$$\begin{aligned} \Pi \psi_{\nu\rho}(x) &= \frac{\Gamma^s(1+a)}{\Pi a_\nu^a} \frac{\Pi S_{\nu\rho}}{q^s} (1-\rho^{-1}x)^{-sa} \\ &= \frac{\Gamma^s(1+a)}{\Pi a_\nu^a} \frac{\Pi S_{\nu\rho}}{q^s} \sum_{m=0}^{\infty} \frac{sa \dots (sa+m-1)}{m!} \rho^{-m} x^m; \end{aligned}$$

whence

$$\begin{aligned} & \frac{1}{2\pi i} \int_C \Pi \psi_{\nu\rho}(x) \frac{dx}{x^{n+1}} \\ &= \text{coef. of } x^n \text{ above} = \frac{\Gamma^s(1+a)}{\Pi a_\nu^a} \frac{\Pi S_{\nu\rho}}{q^s} \frac{sa \dots (sa+n-1)}{n!} \rho^{-1} \\ &= \frac{\Gamma^s(1+a)}{\Gamma(sa) \Pi a_\nu^a} \frac{\Pi S_{\nu\rho}}{q^s} \frac{\Gamma(sa+n)}{n!} \rho^{-n}. \end{aligned}$$

Hence, if we use the notation

$$A_0(q) = q^{-s} \sum_{\rho(q)} \Pi S_{\nu\rho} \rho^{-n},$$

and remember that on major arcs $q \leq n^a$, (42) becomes

$$(43) \quad \left| r(n) - \frac{\Gamma(sa+n)}{n!} \left(\frac{\Gamma^s(1+a)}{\Gamma(sa) \Pi a_\nu^a} \sum_{q \leq n^a} A_0(q) \right) \right| < b_{12} n^{sa-1-c_{35}}.$$

Next we propose to replace $\Gamma(sa+n)/n!$ in (43) by n^{sa-1} . By Lemma 9, with $\beta = sa-1$, we have

$$\left| \frac{\Gamma(sa+n)}{n!} - n^{sa-1} \right| < b_{13} n^{sa-2};$$

and hence, by Lemma 5 (a),

$$\begin{aligned} (44) \quad & \left| \frac{\Gamma(sa+n)}{n!} \left(\frac{\Gamma^s(1+a)}{\Gamma(sa) \Pi a_\nu^a} \sum_{q \leq n^a} A_0(q) \right) - n^{sa-1} \left(\frac{\Gamma^s(1+a)}{\Gamma(sa) \Pi a_\nu^a} \sum_{q \leq n^a} A_0(q) \right) \right| \\ & < b_{14} n^{sa-2} \sum_{q=1}^{\infty} q^{-1-1/2K} = b_{15} n^{sa-2}. \end{aligned}$$

Taken together, (43) and (44) yield

$$(45) \quad \left| r(n) - n^{sa-1} \frac{\Gamma^s(1+a)}{\Gamma(sa) \Pi a_\nu^a} \sum_{q \leq n^a} A_0(q) \right| < b_{16} n^{sa-1-c_{36}}.$$

We now replace the sum in (45) by

$$\mathfrak{S}_0 = \sum_{q=1}^{\infty} A_0(q).$$

By Lemma 5 (a),

$$\left| \sum_{q > n^a} A_0(q) \right| \leq \sum_{q > n^a} b_6 q^{-1-1/2K} < b_{17} n^{-c_{37}};$$

so that (45) becomes

$$(26) \quad \left| r(n) - \frac{\Gamma^s(1+a)}{\Gamma(sa) \prod a_\nu^a} n^{sa-1} \mathfrak{S}_0 \right| < b_7 n^{sa-1-c_{27}},$$

completing the proof of Theorem 1.

The importance of Theorem 1 depends upon a positive sign for $\mathfrak{S}_0$, as will become evident at the end of Chapter II.

CHAPTER II.

The second Hardy-Littlewood theorem extended.

The aim of this chapter, which culminates in Theorem 2, is to find sufficient conditions to be imposed on s and on the coefficients a_ν to ensure that $\mathfrak{S}_0 > 0$.

8. *Expression of $\mathfrak{S}_0$ as a function of the χ_{0p} .* If we define

$$\chi_{0p} = 1 + A_0(p) + A_0(p^2) + \dots = \sum_{l=0}^{\infty} A_0(p^l), \quad p \text{ prime},$$

then a straightforward generalization of the proofs of Satz 280–Satz 284 in Landau, with $\mathfrak{S}$ replaced by $\mathfrak{S}_0$, $A(q)$ by $A_0(q)$, S_p by $S_{\nu p}$, and χ_p by χ_{0p} , establishes

LEMMA 16. *If for a fixed s the singular series $\mathfrak{S}_0$ converges absolutely for every n , then the series χ_{0p} converges absolutely for every n and*

$$\mathfrak{S}_0 = \prod_p \chi_{0p}.$$

9. *A lower bound for χ_{0p} for all p 's.* Consider, for integral $l \geq 0$, the congruence

$$(46) \quad \sum_{\nu=1}^s a_\nu h_\nu^k \equiv n \pmod{p^l}, \quad 0 \leq h_\nu < p^l.$$

We read $p+q$ as “ p not a divisor of q ” and define

$M_0(p^l, n)$ = total number of solutions of (46);

$N_0(p^l, n)$ = number of solutions of (46) with $p + \text{g.c.d.}(h_\nu)$;

$N_{sp}(p^l, n)$ = number of *special solutions* of (46); i.e., the number of solutions having $p + \text{g.c.d.}(a_\nu, h_\nu)$.

With these definitions the methods of Landau's Satz 285 and Satz 286, applied to $A_0(p^l)$ and $M_0(p^l)$ instead of to $A(p^l)$ and $M(p^l)$, yield an easy proof of the following Lemma.

LEMMA 17. For $l \geq 0$,

$$\sum_{\lambda=0}^l A_0(p^\lambda) = p^{-(s-1)l} M_0(p^l).$$

We define four new symbols:

Θ = multiplicity of p in k .

$$k_0 = k/p^\Theta.$$

$$\gamma = \Theta + 1 \text{ if } p > 2.$$

$$= \Theta + 2 \text{ if } p = 2.$$

$$P = p^\gamma.$$

It follows, Landau, 282, that in all instances $\gamma \leq k$. Moreover, we have

LEMMA 18. (Landau, Satz 290.) Let $l \geq \gamma$; let x, y, z be integers for which

$$x = y + zp^{l-\Theta-1}.$$

Then

$$x^k \equiv y^k + k_0 y^{k-1} z p^{l-1} \pmod{p^l}.$$

LEMMA 19. For $l > \gamma$,

$$N_{sp}(p^l) = p^{s-1} N_{sp}(p^{l-1}).$$

Recall that $N_{sp}(p^l)$ denotes the number of solutions of

$$(47) \quad \sum_{\nu=1}^s a_\nu h_\nu^k \equiv n \pmod{p^l}, \quad 0 \leq h_\nu < p^l, \quad p + \text{g.c.d.}(a_\nu h_\nu).$$

The admissible h_ν are given without duplication by

$$(48) \quad h_\nu = y_\nu + z_\nu p^{l-\Theta-1}, \quad 0 \leq y_\nu < p^{l-\Theta-1}, \quad 0 \leq z_\nu < p^{\Theta+1}, \quad p + \text{g.c.d.}(a_\nu h_\nu).$$

By Lemma 18, then,

$$h_\nu^k \equiv y_\nu^k + k_0 y_\nu^{k-1} z_\nu p^{l-1} \pmod{p^l},$$

so that (47) becomes

$$\sum_{\nu=1}^s a_\nu (y_\nu^k + k_0 y_\nu^{k-1} z_\nu p^{l-1}) \equiv n \pmod{p^l}$$

with the conditions (48); and this in turn is equivalent to the pair of congruences

$$(49) \quad \sum_{\nu=1}^s a_\nu y_\nu^k \equiv n \pmod{p^{l-1}}, \quad 0 \leq y_\nu < p^{l-\Theta-1}, \quad p + \text{g.c.d.}(a_\nu y_\nu),$$

$$(50) \quad k_0 \sum_{\nu=1}^s a_\nu y_\nu^{k-1} z_\nu \equiv \left(n - \sum_{\nu} a_\nu y_\nu^k \right) / p^{l-1} \pmod{p}, \quad 0 \leq z_\nu < p^{\Theta+1}.$$

Congruence (49) would have $N_{sp}(p^{l-1})$ special solutions for y_ν ranging from zero to p^{l-1} ; but by Lemma 18, taken modulo p^{l-1} , the congruence

$$y \equiv u \pmod{p^{l-\Theta-1}}$$

implies that

$$y^k \equiv u^k \pmod{p^{l-1}};$$

so that residual sets modulo $p^{l-\Theta-1}$ satisfy (49). Hence (49) actually has $p^{-\Theta s} N_{sp}(p^{l-1})$ special solutions.

For every special solution of (49) there are $p^{(\Theta+1)s-1}$ sets of z_ν satisfying (50); for p cannot divide every product $a_\nu y_\nu$. If, for example, $p \nmid a_1 y_1$, we assign arbitrarily $z_2, \dots, z_s$ (in $p^{\Theta+1}$ ways for each z , in $p^{(\Theta+1)(s-1)}$ ways for the set $z_2, \dots, z_s$), and z_1 is then uniquely determined mod p (since $p \nmid k_0 a_1 y_1^k$) by (50). To this determination of $z_1 \pmod{p}$ there correspond p^Θ values on $0 \leq z_1 < p^{\Theta+1}$, and hence, for each special solution of (49), there are $p^{(\Theta+1)(s-1)} p^\Theta = p^{(\Theta+1)s-1}$ special solutions of (50). Since there are $p^{-\Theta s} N_{sp}(p^{l-1})$ of these, we conclude that

$$N_{sp}(p^l) = p^{-\Theta s} N_{sp}(p^{l-1}) p^{(\Theta+1)s-1} = p^{s-1} N_{sp}(p^{l-1}).$$

When $p > \max a_\nu$, primitive and special solutions coincide, giving, as a corollary to Lemma 19, the following

LEMMA 20. For $l > \gamma$ and $p > \max a_\nu$,

$$N_0(p^l) = p^{s-1} N_0(p^{l-1}).$$

The next two lemmas are obvious for $l = \gamma$. For $l > \gamma$ they follow from Lemmas 19 and 20 applied $l - \gamma$ times.

LEMMA 21. For $l \geq \gamma$,

$$N_{sp}(p^l) = p^{(l-\gamma)(s-1)} N_{sp}(P).$$

LEMMA 22. For $l \geq \gamma$ and for $p > \max a_\nu$,

$$N_0(p^l) = p^{(l-\gamma)(s-1)} N_0(P).$$

LEMMA 23. Let the multiplicity of p^k in n , $n \neq 0$, be β , i.e. $n = (p^k)^\beta p^\sigma$, where $0 \leq \sigma < k$; and let

$$l_0 = \max(\beta k + \sigma + 1, \beta k + \gamma).$$

Then, if $p > \max a_\nu$,

$$(51) \quad A_0(p^l) = 0 \quad \text{for } l > l_0,$$

and

$$(52) \quad \chi_{0p} = P^{1-s} N_0(P, 0) \sum_{\alpha=0}^{\beta-1} p^{\alpha(k-s)} + p^{\beta(k-s)} P^{1-s} N_0(P, n/p^{\beta k}).$$

The proof is strictly parallel to that of Satz 293, Landau, with A_0 , M_0 , N_0 , and χ_{0p} used instead of A , M , N , and χ_p , and Lemmas 17 and 22 instead of Satz 286 and Satz 292, respectively.

In particular, if in Lemma 23 we have $p+2kn$, then equations (51) and $l_0=1$ imply that

$$(53) \quad \chi_{0p} = 1 + A_0(p).$$

Lemma 23 finds its chief application in Section 10. Lemma 21 enables us to prove the final lemma of this section:

LEMMA 24. *If, for every n ,*

$$N_{sp}(P, n) > 0,$$

then, for every n , $\chi_{0p} \geq P^{1-s} = b(p)$.

Proof. By Lemma 21, for $l \geq \gamma$,

$$(54) \quad p^{-(s-1)l} N_{sp}(p^l) = p^{-\gamma(s-1)} N_{sp}(P) = P^{1-s} N_{sp}(P).$$

By definition, Lemma 17, and (54),

$$\begin{aligned} \chi_{0p} &= \lim_{l \rightarrow \infty} \sum_{\lambda=0}^l A_0(p^\lambda) = \lim_{l \rightarrow \infty} p^{-(s-1)l} M_0(p^l) \\ &\geq \varliminf_{l \rightarrow \infty} p^{-(s-1)l} N_{sp}(p^l) = \lim_{l \rightarrow \infty} P^{1-s} N_{sp}(P), \end{aligned}$$

which cannot be less than P^{1-s} if the congruence (46) has a special solution (mod P) for every n .

10. *A lower bound for χ_{0p} for large p 's.* Most of the proofs of this section are at once reduced to cases in Landau by the hypothesis

$$(H) \quad p > \max a_\nu,$$

which implies that $(p, a_\nu) = 1$; so that, if ρ_1 is a primitive q -th root of unity, so is $\rho_1^{a_\nu} = \rho_2$, and we have $S_{\nu\rho_1} = S_{\rho_2}$. Where no other difficulty than this is involved, the only proof offered is a reference to the analogue in Landau.

LEMMA 25. *If $p > \max a_\nu$, $q = p^l$, $p+k$, $2 \leq l \leq k$, then*

$$S_{\nu p} = p^{l-1}.$$

The proof reduces by (H) to Satz 307, Landau.

LEMMA 26. *For $p > \max a_\nu$ and $q = p$,*

$$|S_{\nu p}| \leq (k-1) \sqrt{p}.$$

See (H) and Satz 311, Landau.

LEMMA 27. For $p > \max a_\nu$, $q = p^l$, $l > k$,

$$S_{\nu p} = p^{k-1} S_{\nu p^k}.$$

See (H) and Satz 312, Landau.

LEMMA 28. Let $T_{\nu p} = S_{\nu p}/q^{1-a}$. Then, for $p > \max a_\nu$, $q = p^l$, $l > k$,

$$T_{\nu p} = T_{\nu p^k}.$$

See (H) and Satz 313, Landau.

LEMMA 29. For $q = p^l$ and $l \geq 1$,

$$\begin{aligned} |T_{\nu p}| &\leq 1 \quad \text{for } p > c_{38}, \\ |T_{\nu p}| &\leq c_{39} \quad \text{for all values of } p. \end{aligned}$$

See the proof of Satz 314, Landau, using Lemma 28 instead of Satz 313. Corresponding to the cases (1), (2), and (3), take

(1) $p > k + \max a_\nu$, $2 \leq l \leq k$, Lemma 25 for Satz 307.

(2) $p > k + \max a_\nu$, $l = 1$, Lemma 27 for Satz 311.

(3) $p \leq k + \max a_\nu$, with $c_{41} = k + \max a_\nu$.

LEMMA 30. For all values of q ,

$$|T_{\nu p}| < c_{15} \quad \text{and} \quad |S_{\nu p}| < c_{15} q^{1-a}.$$

See (H) and Satz 315, Landau, with Lemma 29 instead of Satz 314.

LEMMA 31. $|A_0(q)| < b_{19} q^{1-sa}$.

The proof is by Lemma 30; for

$$|A_0(q)| \leq \sum_{\rho(q)} \left| \prod_{\nu=1}^s S_{\nu p} \right| q^{-s} < q(c_{15} q^{-a})^s = b_{19} q^{1-sa}.$$

By Lemma 31 it follows that, for $s > 2k$, the singular series

$$\mathfrak{S}_0 = \sum_{q=1}^{\infty} A_0(q)$$

converges absolutely, and hence, by Lemma 16,

$$(55) \quad \mathfrak{S}_0 = \prod_p \chi_{0p}$$

is valid for $s > 2k$. Since we consider only $k > 2$, $s > (k-2)K+4$ implies that $s > 2k$, and (55) holds for the s of Theorem 1.

LEMMA 32. $|A_0(p)| < b_{21} p^{1-\frac{1}{2}s}.$

The proof has two cases. For $p \leq \max a_\nu$,

$$|A_0(p)| \leq p^{-s} \sum_{\rho(p)} \left| \prod_{\nu=1}^s S_{\nu\rho} \rho^{-n} \right| \leq p^{-s} (p-1) p^s \cdot 1 = p-1 < \max a_\nu.$$

For $p > \max a_\nu$ Lemma 26 gives

$$|A_0(p)| = \left| \sum_{\rho(p)} p^{-s} \prod_{\nu=1}^s S_{\nu\rho} \rho^{-n} \right| < p[p^{-1}(k-1)\sqrt{p}]^s = b_{22} p^{1-\frac{1}{2}s}.$$

LEMMA 33. For $p > k + \max a_\nu$ and $p \nmid n$,

$$|\chi_{0p} - 1| = |A_0(p)| < b_{21} p^{1-\frac{1}{2}s}.$$

Since $p \nmid 2kn$, the equality is implied by (53) and the inequality by Lemma 32.

LEMMA 34. For $p > k + \max a_\nu$ and $p^k \nmid n$,

$$|\chi_{0p} - 1| < b_{25} p^{1-\frac{1}{2}s}.$$

The proof is parallel to that of Satz 321, Landau, with χ_{0p} , $A_0(p)$, $S_{\nu\rho}$ for χ_p , $A(p)$, S_ρ , respectively, Lemma 23 for Satz 293, Lemma 33 for Satz 320, Lemma 32 for Satz 318, and Lemma 25 for Satz 307. The case $p \mid 2k$ is vacuous because we have $p > k + \max a_\nu$.

LEMMA 35. $\chi_{0p} > 1 - b_{28} p^{1-\frac{1}{2}s}.$

The case $p^k \nmid n$ is provided for by Lemma 34. The case $p^k \mid n$ implies that $\beta > 0$. If, then,

(1) $p > k + \max a_\nu$, $\gamma = 1$, $P = p^\gamma = p$, and, by Lemma 23,

$$\chi_{0p}(n) \geq p^{1-s} N_0(p, 0) = p^{1-s} N_0(p, p) = \chi_{0p}(p) > 1 - b_{25} p^{1-\frac{1}{2}s},$$

where the final inequality follows from Lemma 34.

(2) $p \leq k + \max a_\nu$, for an appropriate b_{29} ,

$$\chi_{0p} \geq 0 > 1 - b_{29} p^{1-\frac{1}{2}s}.$$

As an immediate corollary, Lemma 35 yields

LEMMA 36. For $s \geq 5$,

$$\chi_{0p} > 1 - b_{28} p^{-\frac{3}{2}}.$$

Lemma 36 in turn implies that

LEMMA 37. For $s \geq 5$ and $p > b_{18}$,

$$\chi_{0p} > 1 - p^{-\frac{5}{2}}.$$

11. *The proof of Theorem 2.* We can now prove the extension of what Landau calls the second Hardy-Littlewood theorem.

THEOREM 2. *If $s \geq (k-2)K+5$, and if for every n there is a special solution of*

$$\sum_{\nu=1}^s a_{\nu} h_{\nu}^k \equiv n \pmod{P},$$

then

$$\mathfrak{S}_0 > b_4.$$

Proof. By Lemmas 31, 24, and 37,

$$\mathfrak{S}_0 = \prod_p \chi_{0p} = \prod_{p \leq b_{18}} \chi_{0p} \prod_{p > b_{18}} \chi_{0p} > \prod_{p \leq b_{18}} b(p) \prod_{p > b_{18}} (1-p^{-\frac{s}{2}}) = b_4.$$

The hypotheses of Theorems 1 and 2 always yield a Waring theorem of the type which we seek, for Theorem 1 implies that

$$\lim_{n \rightarrow \infty} \left| \frac{r(n)}{n^{sa-1}} - \frac{\Gamma^s(1+a)}{\Gamma(sa) \prod a_{\nu}^a} \mathfrak{S}_0 \right| = 0.$$

Hence, for $n > b_1$,

$$\begin{aligned} \left| \frac{r(n)}{n^{sa-1}} - \frac{\Gamma^s(1+a)}{\Gamma(sa) \prod a_{\nu}^a} \mathfrak{S}_0 \right| &< \frac{b_4}{2} \frac{\Gamma^s(1+a)}{\Gamma(sa) \prod a_{\nu}^a}, \\ \frac{r(n)}{n^{sa-1}} &> \left| \frac{\Gamma^s(1+a)}{\Gamma(sa) \prod a_{\nu}^a} \left(\mathfrak{S}_0 - \frac{b_4}{2} \right) \right| > \frac{b_4}{2} \frac{\Gamma^s(1+a)}{\Gamma(sa) \prod a_{\nu}^a} = b_2, \end{aligned}$$

i.e., for n sufficiently large, $r(n) > 0$. Our problem is thus reduced to finding when the second hypothesis of Theorem 2 is satisfied.

We summarize the implications of Theorems 1 and 2 in

THEOREM 2'. *Under the hypotheses of Theorem 2 there exists a positive constant b_1 , depending only on k , s , and $\max a_{\nu}$, such that, for every integer $n > b_1$, there is a solution of the diophantine equation*

$$\sum_{\nu=1}^s a_{\nu} h_{\nu}^k \equiv n \pmod{P} \quad (h_{\nu} \geq 0).$$

CHAPTER III.

Sufficient conditions for Waring sets of coefficients.

Henceforth we shall keep $s \geq (k-2)K+5$, as required in Theorems 1 and 2, and seek sets of coefficients a_{ν} for which the congruence

$$(55') \quad \sum_{\nu=1}^s a_{\nu} h_{\nu}^k \equiv n \pmod{P}$$

has a special solution for every integer n and every prime p . A special solution, let us recall, must have at least one of the products $a_\nu h_\nu$ prime to p . Such a set of coefficients, since it always yields an asymptotic Waring theorem of our generalized type, will be called an *admissible set* or a *Waring set*.

12. *The primitivity conditions.* The residue class represented by $n_1 \pmod{m}$ is called *equivalent* to the residue class represented by n_2 if the congruence

$$x^k n_1 \equiv n_2 \pmod{m}$$

has a solution with $(x, m) = 1$. It is shown in Landau, 288, that equivalence is reflexive, symmetric, and transitive, and furthermore that, for $p > 2$, the number of sets of equivalent residue classes mod P is $1 + r$, where

$$(55'') \quad r = \frac{P-1}{p-1} (k, p-1),$$

and that the set represented by zero contains no other residue class, but that every other set has $(p-1)/(k, p-1)$ incongruent equivalent residue classes.

LEMMA 38. *If, for a given prime p , the coefficients $a_1, \dots, a_r$ are all prime to p , then, for every $n \not\equiv 0 \pmod{P}$, there is a special solution of*

$$(56) \quad \sum_{\nu=1}^r a_\nu h_\nu^k \equiv n \pmod{P}.$$

Note that it is sufficient to prove the lemma for a single representative from each of the r sets of equivalent residue classes. For if $(h_1, \dots, h_r)$ is a special solution of (56) for $n = n_1$, and if n_2 is equivalent to n_1 , then there is an x prime to p such that

$$(57) \quad \hat{n}_2 \equiv x^k n_1 \equiv x^k \sum_{\nu=1}^r a_\nu h_\nu^k \equiv \sum_{\nu=1}^r a_\nu (x h_\nu)^k \pmod{P},$$

thus exhibiting a special solution for $n = n_2$.

Let any set of equivalent classes containing an a_ν be called an A -set. If $n = a_j$, then $(0, 0, \dots, h_j = 1, \dots, 0, 0)$ is a special solution of (56). Hence, by (57), for every n in an A -set, there is a special solution of (56). If, therefore, every set is an A -set, the lemma is proved.

If t sets contain no a_ν (*blank sets*), then somewhere among the A -sets are distributed t extra a_ν . Choose in each A -set a particular a_ν as representative and label the t extra coefficients $e_1, e_2, \dots, e_t$. For the

moment let a denote the a_i which represents the A -set containing e_1 . Since $(a, p) = 1$, each of the $P-1$ residue classes over which n ranges may be represented by a number of the form ma , $1 \leq m < P$. Now represent each set by the least multiple ma which it contains and arrange these in ascending order

$$m_1 a = 1 \cdot a, \quad m_2 a, \quad \dots, \quad m_r a.$$

Let $m_j a$ represent the first blank set. Since e_1 is equivalent to a , there is an x prime to p for which $e_1 x^k \equiv a \pmod{P}$, and hence

$$m_j a - e_1 x^k \equiv m_j a - a \equiv (m_j - 1)a \pmod{P}.$$

Since $m_j \geq 2$, we have $a \leq (m_j - 1)a < m_j a$; therefore $(m_j - 1)a$ belongs to one of the $j-1$ preceding A -sets. Hence there is a special solution of the partial congruences

$$m_j a - e_1 x^k \equiv \sum_{a_i \neq e_1, \dots, e_t} a_i h_i^k, \quad m_j a \equiv \sum_{a_i \neq e_2, \dots, e_t} a_i h_i^k \pmod{P}.$$

If e_2 is in a different set from e_1 , we represent the residue classes by multiples of a new a , rearrange the sets, and repeat the above argument. By using in succession the t extra coefficients, we deal with all the t blank sets.

If $p = 2$, then $\gamma = \Theta + 2$, $P = 2^{\Theta+2}$, $r = 2^{\Theta+2} - 1 =$ the number of residue classes over which n ranges $\geq$ the number of sets of equivalent residue classes. Hence the preceding proof is applicable, possibly with some leeway, to the case $p = 2$, as well as to $p > 2$.

LEMMA 39. *If, for a given prime p , the coefficients $a_1, \dots, a_{r+1}$ are all prime to p , then for every n there is a special solution of*

$$\sum_{v=1}^{r+1} a_v h_v^k \equiv n \pmod{P}.$$

If $n \not\equiv 0 \pmod{P}$, the proof follows from Lemma 38. If $n \equiv 0 \pmod{P}$, then $n - a_{r+1} \not\equiv 0$, and again, by Lemma 38,

$$n \equiv (n - a_{r+1}) + a_{r+1} \equiv \sum_{v=1}^r a_v h_v^k + a_{r+1} \cdot 1^k.$$

Lemma 39 yields at once

THEOREM 3. *A set of coefficients a_i is admissible if for every prime p at least $r+1$ of the a_i are prime to p .*

Since, by the computations given by Landau on p. 291, r never exceeds $4k-1$, Theorem 3 gives

THEOREM 4. *A set of coefficients a_i is admissible if for every prime p at least $4k$ of the a_i are prime to p .*

COROLLARY 1. *By taking $4k$ of the a_v equal to 1, the rest may be left arbitrary positive integers.*

COROLLARY 2. *By taking $4k+1$ of the a_v relatively prime in pairs, the rest may be left arbitrary.*

Our theorems will gain in conciseness if we adopt the following definition: *A set a_v is of primitivity m if for every prime p at least m of the a_v are prime to p .* For example, Theorem 4 would then read "A set of coefficients a_v is admissible if it is of primitivity $4k$ ".

The condition "primitivity $4k$ " of Theorem 4 will be improved in the following sections for certain exponents k , but it should be noted that there are exponents for which no better sufficiency theorem, in terms only of the primitivity of the coefficients, can be obtained. For $k=2^\lambda$, for example, the coefficients of the diophantine equation

$$\sum_{\nu=1}^{4 \cdot 2^\lambda - 1} h_\nu^{2^\lambda} + \sum_{\mu=4 \cdot 2^\lambda}^8 2^{2^\lambda} h_\mu^{2^\lambda} = N$$

are of primitivity $4k-1$, and yet this equation has no solution in non-negative integers if $\lambda \geq 3$ and $N \equiv 4 \cdot 2 \pmod{2^{2^\lambda}}$. The proof rests on notes by A. Kempner, *Math. Annalen*, 72 (1912), 395.

13. *The case k an odd prime.* It is easy to verify that, if k is an odd prime,

$$(58) \quad \begin{aligned} (a) & \text{ for } p \neq k, \quad r \leq k, \\ (b) & \text{ for } p = k, \quad r = k+1, \quad P = k^2. \end{aligned}$$

With these values of r we can improve the $4k$ of Theorem 4.

THEOREM 5. *If k is an odd prime, a set of coefficients a_v is admissible if it is of primitivity $k+2$.*

This follows from (58) and Theorem 3.

THEOREM 6. *If k is an odd prime, a set of coefficients a_v is admissible if it is of primitivity $k+1$ and if, for every n , $N_{sp}(k^2, n) > 0$.*

This is proved for (58a) by Theorem 3 and primitivity $k+1$; for (58b) by $N_{sp}(k^2, n) > 0$.

THEOREM 7. *If k is an odd prime, a set of coefficients a_v is admissible if it is of primitivity $k+1$ and if it contains a subset with g.c.d. = 1 for which, with appropriate choice of signs, $\sum \pm a_v = 0$.*

The proof for $n \neq 0$ is by (58) and Lemma 38. For $n \equiv 0$, note that k odd implies that, for all moduli P , $(-1)^k \equiv -1$ and $1^k \equiv 1$; hence $N_{sp}(P, 0) > 0$ by the last hypothesis of Theorem 7.

THEOREM 8. *If k is an odd prime, a set of coefficients a_ν is admissible if (1) it is of primitivity k , if (2) for every n , $N_{sp}(k^2, n) > 0$, and if (3) it contains a subset with g.c.d. = 1 for which, with appropriate choice of signs, $\Sigma \pm a = 0$.*

For $n \equiv 0$, the proof is as in Theorem 7. For $n \neq 0$, it follows for (58a) from (1) and Lemma 38; for (58b), from (2).

For particular exponents better theorems are obtainable. Some of these are developed for $k = 3, 4$, and 5 in Chapter IV.

14. *Ordinary and special base sets.* We shall develop a method, particularly fruitful for odd exponents, of constructing certain admissible sets of coefficients not obtainable by Theorems 3–8.

We first define a *base set of order r* as a set $a_1, \dots, a_r$ having

$$(59) \quad \begin{cases} (a) \text{ for } k \text{ even,} \\ \quad a_1 = 1, \quad a_j \leq a_{j+1} \leq 1 + \sum_{\nu=1}^j a_\nu, \\ (b) \text{ for } k \text{ odd,} \\ \quad a_1 = 1, \quad a_j \leq a_{j+1} \leq 1 + 2 \sum_{\nu=1}^j a_\nu, \end{cases}$$

and a *special base set of order r* as having

$$(60) \quad \begin{cases} (a) \text{ for } k \text{ even,} \\ \quad a_1 = a_2 = 1, \quad a_j \leq a_{j+1} \leq \sum_{\nu=1}^j a_\nu \text{ for } j \geq 2, \\ (b) \text{ for } k \text{ odd,} \\ \quad a_1 = a_2 = 1, \quad a_j \leq a_{j+1} \leq 1 + 2 \sum_{\nu=1}^j a_\nu \text{ for } j \geq 2. \end{cases}$$

These sets range respectively from

$$1, 1, \dots, 1_r \text{ to } 1, 2, 2^2, \dots, 2^{r-1}.$$

$$1, 1, \dots, 1_r \text{ to } 1, 3, 3^2, \dots, 3^{r-1}.$$

$$1, 1, \dots, 1_r \text{ to } 1, 1, 2, \dots, 2^{r-2}.$$

$$1, 1, \dots, 1_r \text{ to } 1, 1, 5, 5.3, \dots, 5.3^{r-3}.$$

From the above definitions follows Lemma 40 (41), obviously for k even and for k odd by the fact that -1 is then a k -ic residue mod P for every prime p .

LEMMA 40 (41). *If $a_1, \dots, a_r$ is a (special) base set of order r , then, for $1 \leq n \leq a_j$, there is a (special) solution of the partial congruence*

$$n \equiv \sum_{\nu=1}^j a_\nu h_\nu^k \pmod{P}.$$

We can now prove the following double lemma.

LEMMA 42 (43). *If $a_1, \dots, a_r$ is a (special) base set of order r , with r as in (55''), then, for every $n \equiv 0 \pmod{P}$, there is a (special) solution of*

$$n = \sum_{\nu=1}^r a_\nu h_\nu^k \pmod{P}.$$

The proof of Lemma 42 (43) resembles that of Lemma 38. For $p > 2$, let each of the r sets of equivalent residue classes mod P over which n ranges be represented by the least positive residue which it contains and arrange these in ascending order

$$1 = n_1, n_2, \dots, n_r.$$

As in Lemma 38, it is sufficient to make a proof for a single representative from each set. Since $n_1 = a_1 = 1$, there is a (special) solution of

$$n_1 \equiv a_1 h_1^k \pmod{P}$$

with $h_1 = 1$. If $n_2 \leq a_2$, there is a (special) solution of

$$n_2 \equiv \sum_{\nu=1}^2 a_\nu h_\nu^k \pmod{P}$$

by Lemma 40 (41). If $n_2 > a_2$, then $0 < n_2 - a_2 < n_2$, so that $n_2 - a_2$ belongs to the first set and there are (special) solutions for

$$n_2 - a_2 \cdot 1^k \equiv a_1 h_1^k, \quad n_2 \equiv \sum_{\nu=1}^2 a_\nu h_\nu^k \pmod{P}.$$

An induction completes the proof for $p > 2$.

For $p = 2$, the closing paragraph under Lemma 38 applies.

THEOREM 9. *If for a given prime p the coefficients a_ν contain an ordinary base set of order r , and if the remaining coefficients have g.c.d. = 1, then, for every n ,*

$$N_{sp}(P, n) > 0.$$

Proof. For $p \nmid n$ there is a solution by Lemma 42. It is necessarily special. For $p \mid n$ one of the extra coefficients a_λ is prime to p , so that

$p + n - a_\lambda$. Hence, by Lemma 42, there are special solutions of

$$n - a_\lambda \cdot 1^k \equiv \sum_{\nu=1}^r a_\nu h_\nu^k, \quad n \equiv \sum_{\nu=1}^s a_\nu h_\nu^k \pmod{P}.$$

THEOREM 10. *If k is odd and if the coefficients a_ν contain a special base set of order r , then, for every n ,*

$$N_{sp}(P, n) > 0.$$

The proof is by Lemma 43 for $n \not\equiv 0 \pmod{P}$. For $n \equiv 0 \pmod{P}$, since $a_1 = a_2 = 1$, there is a special solution with $h_1 = 1$, $h_2 = -1$, remaining h 's zero.

THEOREM 11. *If k is even and if the coefficients a_ν contain a special base set of order $r+1$, then, for every n ,*

$$N_{sp}(P, n) > 0.$$

Lemma 43 proves the case $n \not\equiv 0 \pmod{P}$. For $n \equiv 0$, note that, by dropping off a_1 , any special base set of order $r+1$ becomes an ordinary base set of order r . By counting $a_1 = 1$ as one of the "remaining coefficients", Theorem 9 applies.

15. Derived coefficients. Whenever k is odd, it is possible to assign most of the coefficients and to determine the rest of them as functions of these. While the process is quite general and a description of it properly belongs here, it seems preferable to illustrate it instead for a special case. This is done in § 17.

CHAPTER IV.

Special cases.

It is interesting to inquire what the theorems of Chapter III yield when applied to cubes, fourth powers, and fifth powers.

16. The case $k = 3$. For $k = 3$, Theorem 1 requires $s \geq 9$; if $p = 3$, $r = 4$; if $p \neq 3$, $r \leq 3$. Hence the theorems of § 13 reduce to:

THEOREM 12. *For $k = 3$, any set of nine (or more) coefficients a_ν is admissible if it satisfies one of the following sets of conditions:*

(a) *Primitivity 5.* (By Theorem 5.)

(b) *Primitivity 4; and some subset of the a_ν has g.c.d. = 1 and satisfies $\Sigma \pm a_\nu = 0$. (By Theorem 7.)*

Example: 5, 7, 12, 13, 19, +4 arbitrary.

(c) *Primitivity 4; and for every n there is a special solution of $n \equiv \sum_{\nu=1}^9 a_\nu h_\nu^3 \pmod{9}$. (By Theorem 6.)*

Example: 29, 31, 37, 41, 53, +4 arbitrary.

(d) *Primitivity 3; for every n there is a special solution of*

$$n \equiv \sum_{\nu=1}^9 a_\nu h_\nu^3 \pmod{9};$$

and some subset of the a_ν has g.c.d. = 1 and satisfies $\Sigma \pm a_\nu = 0$ (By Theorem 8.)

Example: 12, 19, 31, 53, +5 arbitrary.

By Theorems 9 and 10 when $p \neq 3$, and by verifying that $N_{sp}(9, n) > 0$ for every n when $p = 3$, we can prove

THEOREM 13. *Nine coefficients a_ν are admissible if their respective values are given by the following table:*

a_1, a_2, a_3	$a_4, \dots, a_9$	Proof by
1 1 1	$a_4 \not\equiv 0 \pmod{9}$, rest arbitrary	Theorem 10
1 1 2 1 1 3 1 1 4 1 1 5	arbitrary	Theorem 10
1 2 2 · · · 1 2 7 1 3 3 · · · 1 3 9	arbitrary except for g.c.d. = 1	Theorem 9

17. *Derived coefficients for $k = 3$.* Derived coefficients a_1, a_2, a_3 may be obtained as functions of arbitrarily assigned $a_4, \dots, a_9$ by noting that ± 1 are both cubic residues for all moduli P and using any special base set as

follows to determine a_1, a_2, a_3 . With the set 1, 1, 2, for example, consider the positive values of a_1, a_2, a_3 given by

$$(A) \quad a_1 = 1 + (\pm a_4 \pm a_5 \pm a_6), \quad a_2 = 1 + (\pm a_7 \pm a_8), \quad a_3 = 2 + (\pm a_9)$$

when all combinations of $\pm$ signs are used. At least four, two, and one positive values are determined for a_1, a_2 , and a_3 , respectively. This gives eight sets a_1, a_2, a_3 , each of which taken with the original $a_4, \dots, a_9$ constitutes a Waring set. By permuting either the 1, 1, 2 or the $a_4, \dots, a_9$, or both, new sets a_1, a_2, a_3 are found. Moreover, any of the special base sets 1, 1, 3; 1, 1, 4; or 1, 1, 5, or any other partition of $a_4, \dots, a_9$ into three parentheses, will yield further derived values for a_1, a_2, a_3 .

If the coefficients $a_4, \dots, a_9$ are partitioned into four parentheses instead of three, a_1, a_2, a_3 may be determined as above from any three parentheses, while the a 's in the fourth may be regarded as absolutely arbitrary coefficients. It is possible, for example, to assign three coefficients a_4, a_5, a_6 , to determine three more a_1, a_2, a_3 by the equations

$$a_1 = 1 + a_4, \quad a_2 = 1 + a_5, \quad a_3 = 2 \pm a_6,$$

and to leave a_7, a_8, a_9 quite arbitrary.

THEOREM 14. *If to six of the nine coefficients any assigned values are given, the other three coefficients may be determined as functions of these in various ways by equations of the type (A) above.*

18. *The case $k = 4$.* For $k = 4$, Theorem 1 requires $s \geq 21$. Equation (55'') yields

$$r \leq 4 \quad \text{if } p \neq 2, \quad r = 15 \quad \text{if } p = 2.$$

Hence Theorems 3 and 4 reduce to

THEOREM 15. *For $k = 4$, a set of twenty-one (or more) coefficients a_i is admissible*

(a) *if it is of primitivity 16 (by Theorem 4),*

(b) *or if it is of primitivity 5 and if for every n there is a special solution of $n \equiv \sum_{\nu=1}^s a_\nu h_\nu^k \pmod{16}$ (by Theorem 3).*

Example: 17, 49, 67, 37, 43, 71, +15 arbitrary.

THEOREM 16. *For $k = 4$, any set of twenty-one coefficients containing*

$$1, 1, 2, 4, 8$$

is admissible and the remaining sixteen coefficients may be regarded as arbitrary.

Proof. For $p = 2$ we have $P = 16$, and we verify that for every n there is a special solution of the partial congruence

$$h_1^4+h_2^4+2h_3^4+4h_4^4+8h_5^4\equiv n \pmod{P}.$$

Since 1, 1, 2, 4, 8 is a special base set of order 5, and since for $p \neq 2$ we have $r \leq 4$, the case $p \neq 2$ is covered by Theorem 11.

The special base sets of order 5 are 1, 1, 1, 1, 1-4; 1, 1, 1, 2, 2-5; 1, 1, 1, 3, 3-6; 1, 1, 2, 2, 2-6; 1, 1, 2, 3, 3-7; 1, 1, 2, 4, 4-8. Except for the case $p = 2$ the proof of Theorem 16 applies if $a_1, \dots, a_5$ are any of these sets instead of 1, 1, 2, 4, 8. We deal with the case $p = 2$ by specifying enough additional coefficients to ensure a special solution for every n of

$$n \equiv \sum_{\nu=1}^s a_\nu h_\nu^4 \pmod{16}.$$

If permutations are ignored, the admissible sets thus derived can be catalogued numerically in a table of about 1,300 entries. The following is a section of such a table. It is understood that in each entry enough arbitrary coefficients to make $s \geq 21$ must be supplied and that, beginning with a_6 , each coefficient may be replaced by any positive integer to which it is congruent modulo 16.

a	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	$a_{13} \dots a_{20} a_{21}$
1	1	1	1	1	1	4	4	15				
				1		4		6				
				1		4		11				
				1		4		12 15				
				1		4		13 14				
				1		4		13 15	15			
				1		4		14 14	15			
				1		4		14 15	15	15		
				1		4		15 15	15	15	15	
				1		5		5				
				1		5		12				

Theorem 9 shows that the above method of construction yields another class of admissible sets if ordinary base sets ranging from 1, 2, 2, 2, 2 to 1, 2, 4, 8, 16 are used instead of special base sets and if the "arbitrary" coefficients are required to have g.c.d. = 1.

19. The case $k = 5, s \geq 53$. For $p = 5, r = 6$; for $p \neq 5, r \leq 5$.

THEOREM 17. For $k = 5$, a set of fifty-three (or more) coefficients a_ν is admissible

(a) If it is of primitivity 7.

Example: 8 distinct primes + 45 arbitrary.

(b) Or if it is of primitivity 6 and there is a special solution of $n = \sum_{\nu=1}^s a_\nu h_\nu^5$ (mod 25) for every n .

Example: 7 distinct primes + 46 arbitrary.

(c) Or if it is of primitivity 6 and if some subset of the a_ν has g.c.d. = 1 and satisfies $\sum \pm a_\nu = 0$.

Example: 14, 17, 31, 37, 41, 43, 47, + 46 arbitrary.

(d) Or if it is of primitivity 5 and satisfies both (b) and (c).

Examples: 14, 17, 31, 37, 41, 43, + 47 arbitrary.

1, 1, 1, 1, 1, + 48 arbitrary.

THEOREM 18. Of the fifty-three coefficients forty-eight will be arbitrary if the first five are chosen as a special base set of order 5.

There are 890 such sets, with the range 1, 1, 1, 1, 1-9; 1, 1, 1, 1, 2-11; ...; 1, 1, 5, 14, 14-43; 1, 1, 5, 15, 15-45. The theorem follows for $p \neq 5$ by Theorem 10, for $p = 5$ by verification.

THEOREM 19. If the first five coefficients constitute an ordinary base set of order 5, the remaining forty-eight coefficients need have only g.c.d. = 1.

After deleting special base sets, the ordinary base sets of order 5 range from 1, 2, 2, 2, 2 to 1, 3, 9, 27, 81. The existence of a special solution now follows for $p \neq 5$ by Theorem 9, for $p = 5$ by verification.

Note on universal theorems.

The theorems of this paper are proved only for large integers n , but it is interesting to ask which of them hold for all positive integers. If it were not for the forbidding size of the constant b_1 , it would be possible for any particular problem to decide this by actual verification for $1 \leq n \leq b_1$. In any case, the greatest number of exceptions tend to occur for smaller integers. If we can verify that a proposed Waring theorem holds for an initial interval $1 \leq n \leq 1000$, there is a better chance of its being true than if we had inspected $1000000 \leq n \leq 1001000$.

Along with proofs of numerous universal generalizations of Waring's theorem, Prof. L. E. Dickson has given a number of empirical theorems proved for long initial stretches of positive integers*. For $k = 4$, for example, he shows that all positive integers $n \leq 4100$ are represented by the form $(1_{11} 8_1)$, i.e., by a form with eleven coefficients 1 and one coefficient 8. If nine arbitrary coefficients are adjoined to make a total of twenty-one, this is still true, and Theorem 15b adds to the range of n the infinite interval $n > b_1$.

Other forms given by Prof. Dickson and of similar interest are :

for $k = 3$, (11223) , (11224) , (11234) , and (1111114) .

for $k = 4$, $(1_{17} 2_1)$, (1122337) , (1122346) , (11112238) , (11112247) , (11122228) , $(112234m)$ with $m = 6, 7, 8$, or 9 .

Department of Mathematics,

University of Chicago,

Chicago, Illinois, U.S.A.

* L. E. Dickson, *American J. of Math.*, 49 (1927), 241; *American Math. Monthly*, 34 (1927), 177.

VITA.

Ralph E. Huston was born September 16, 1902, in Huntington, Indiana. He was graduated from the Kewanee (Illinois) High School in 1920 and from the University of Chicago in 1923 with the degree of Bachelor of Science. He then spent three years at Merton College, Oxford, as Rhodes Scholar from the State of Illinois, qualifying in 1926 for the degree of Bachelor of Arts in the Final Honour School of English Language and Literature. The following year he studied at the University of Grenoble, and upon returning to America he was for two years an associate professor of modern languages at Southwestern, Memphis, Tennessee.

In the summer of 1929 he resumed graduate study in the Department of Mathematics in the University of Chicago and was in residence for seven consecutive quarters. He taught as a departmental assistant during 1930-31, filled an emergency appointment at Iowa State College for the spring quarter, 1931, and taught again in the University College in the winter of 1932. During 1931-32 he also conducted evening classes for the Extension Division of Indiana University at South Bend, Gary, and East Chicago, Indiana.

He has studied under Professors Barnard, Bartky, Bliss, Coble, Dickson, Graves, Everett, Lane, Laves, Lunn, MacMillan, Moulton, Slaught, Wilczynski, and J. W. A. Young. To all of these he acknowledges his indebtedness, but particularly to Professor Dickson under whose supervision this thesis was written.

